An inner product space is a vector space V equipped with an inner product which is a function $\langle\, ,\, \rangle : V \times V \to \mathbb{R}$ satisfying

(I-a) $\langle \vec{v}, \vec{v} \rangle \geq 0$, and $\langle \vec{v}, \vec{v} \rangle = 0$ iff $v = \vec{0}$,
(I-b) $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$,
(I-c) $\langle \vec{v} + \vec{u}, \vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle + \langle \vec{u}, \vec{w} \rangle$ and $\langle c\vec{v}, \vec{w} \rangle = c\langle \vec{v}, \vec{w} \rangle$.

Theorem 1. *Given an inner product $\langle\, ,\, \rangle$, we can define a norm $\|\ \|$ by*

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}.$$

Namely, $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ satisfies the following

(N-a) $\|\vec{v}\| \geq 0$, *and* $\|\vec{v}\| = 0$ *iff* $v = \vec{0}$,
(N-b) $\|c\vec{v}\| = |c|\|\vec{v}\|$,
(N-c) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

Proof. (I-a) implies (N-a). Also, we can obtain (N-b) by

$$\|c\vec{v}\|^2 = \langle c\vec{v}, c\vec{v} \rangle = c\langle \vec{v}, c\vec{v} \rangle = c\langle c\vec{v}, \vec{v} \rangle = c^2\langle \vec{v}, \vec{v} \rangle = c^2\|\vec{v}\|^2,$$

namely $\|c\vec{v}\| = |c|\|\vec{v}\|$.

Next, we consider two ***unit*** vectors $\vec{a}, \vec{b}$, namely $\|\vec{a}\|^2 = \langle \vec{a}, \vec{a} \rangle = 1$ and $\|\vec{b}\|^2 = \langle \vec{b}, \vec{b} \rangle = 1$. Then, we have

$$\begin{aligned} 0 \leq &\|\vec{a} - \vec{b}\|^2 = \langle \vec{a} - \vec{b}, \vec{a} - \vec{b} \rangle = \langle \vec{a}, \vec{a} - \vec{b} \rangle - \langle \vec{b}, \vec{a} - \vec{b} \rangle \\ =&\langle \vec{a} - \vec{b}, \vec{a} \rangle - \langle \vec{a} - \vec{b}, \vec{b} \rangle = \langle \vec{a}, \vec{a} \rangle - \langle \vec{b}, \vec{a} \rangle - \langle \vec{a}, \vec{b} \rangle + \langle \vec{b}, \vec{b} \rangle \\ =&2 - 2\langle \vec{a}, \vec{b} \rangle, \end{aligned}$$

namely $\langle \vec{a}, \vec{b} \rangle \leq 1$ if $\|\vec{a}\| = \|\vec{b}\| = 1$. We now claim

$$\langle \vec{v}, \vec{w} \rangle \leq \|\vec{v}\|\|\vec{w}\|.$$

If $\vec{v} = \vec{0}$, then $\langle \vec{0}, \vec{w} \rangle = \langle 0\vec{0}, \vec{w} \rangle = 0\langle \vec{0}, \vec{w} \rangle = 0 = \|\vec{0}\|\|\vec{w}\|$. If $\vec{v} \neq 0$ and $\vec{w} \neq 0$, then we can set $\vec{a} = \vec{v}/\|\vec{v}\|$ and $\vec{b} = \vec{w}/\|\vec{w}\|$. Then, we have

$$\begin{aligned} \|\vec{a}\|^2 = \langle \vec{a}, \vec{a} \rangle =&\left\langle \frac{\vec{v}}{\|\vec{v}\|}, \frac{\vec{v}}{\|\vec{v}\|} \right\rangle = \frac{1}{\|\vec{v}\|}\left\langle \vec{v}, \frac{\vec{v}}{\|\vec{v}\|} \right\rangle \\ =&\frac{1}{\|\vec{v}\|}\left\langle \frac{\vec{v}}{\|\vec{v}\|}, \vec{v} \right\rangle = \frac{1}{\|\vec{v}\|^2}\langle \vec{v}, \vec{v} \rangle = 1. \end{aligned}$$

In the same manner, we have $\|\vec{b}\| = 1$. Thus,

$$1 \geq \langle \vec{a}, \vec{b} \rangle = \left\langle \frac{\vec{v}}{\|\vec{v}\|}, \frac{\vec{w}}{\|\vec{w}\|} \right\rangle = \frac{1}{\|\vec{v}\|\|\vec{w}\|}\langle \vec{v}, \vec{w} \rangle,$$

namely $\langle \vec{v}, \vec{w} \rangle \leq \|\vec{v}\|\|\vec{w}\|$. So, we can prove (N-c) as follows

$$\begin{aligned} \|\vec{v} + \vec{w}\|^2 =&\langle \vec{v} + \vec{w}, \vec{v} + \vec{w} \rangle = \langle \vec{v}, \vec{v} \rangle + 2\langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{w} \rangle \\ =&\|\vec{v}\|^2 + 2\langle \vec{v}, \vec{w} \rangle + \|\vec{w}\|^2 \\ \leq&\|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2 = (\|\vec{v}\| + \|\vec{w}\|)^2. \end{aligned}$$

□

Example 2. *There is no inner product* $\langle\ ,\ \rangle_u$ *of* $\mathbb{R}^2$ *such that* $\sqrt{\langle\vec{v},\vec{v}\rangle_w} = \|\vec{v}\|_u$*, where* $\|(v_1,v_2)\|_u = \max\{|v_1|,|v_2|\}$ *is the uniform norm.*

Proof. Assume that there exists an inner product $\langle\ ,\ \rangle_u$ satisfying the condition. We denote $e_1 = (1,0)$ and $e_2 = (0,1)$, Then,

$$1 = \|(1,1)\|^2 = \|e_1 + e_2\|^2 = \|e_1\|^2 + 2\langle e_1, e_2\rangle + \|e_2\|^2 = 2 + 2\langle e_1, e_2\rangle,$$

namely $\langle e_1, e_2\rangle = -\frac{1}{2}$. However,

$$1 = \|(1,-1)\|^2 = \|e_1 - e_2\|^2 = \|e_1\|^2 - 2\langle e_1, e_2\rangle + \|e_2\|^2 = 2 - 2\langle e_1, e_2\rangle,$$

namely $\langle e_1, e_2\rangle = \frac{1}{2}$. Contradiction. □